

Covering groupoids of categorical groups

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Abstract

If X is a topological group, then its fundamental groupoid $\pi_1 X$ is a group-groupoid which is a group object in the category of groupoids. Further if X is a path connected topological group which has a simply connected cover, then the category of covering spaces of X and the category of covering groupoids of $\pi_1 X$ are equivalent. In this paper we prove that if (X, x_0) is an H -group, then the fundamental groupoid $\pi_1 X$ is a categorical group. This enable us to prove that the category of the covering spaces of an H -group (X, x_0) is equivalent to the category of covering groupoid of the categorical group $\pi_1 X$.

Key Words: H -group, covering groupoid, categorical group

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1 Introduction

Covering spaces are studied in algebraic topology, but they have important applications in many other branches of mathematics including differential topology, the theory of topological groups and the theory of Riemann surfaces.

One of the ways of expressing the algebraic content of the theory of covering spaces is using groupoids and the fundamental groupoids. The latter functor gives an equivalence of categories between the category of covering spaces of a reasonably nice space X and the category of groupoid covering morphisms of $\pi_1 X$.

If X is a connected topological group with identity e , $p: (\tilde{X}, \tilde{e}) \rightarrow (X, e)$ is a covering map of pointed spaces such that \tilde{X} is simply connected, then \tilde{X} becomes a topological group with identity \tilde{e} such that p is a morphism of topological groups [5, 10].

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The problem of universal covers of non-connected topological groups was first studied by Taylor in [11]. He proved that a topological group X determines an obstruction class k_X in $H^3(\pi_0(X), \pi_1(X, e))$, and that the vanishing of k_X is a necessary and sufficient condition for the lifting of the group structure to a universal cover. In [8] an analogous algebraic result is given in terms of crossed modules and group objects in the category of groupoids (see also [2] for a revised version, which generalizes these results and shows the relation with the theory of obstructions to extension for groups).

For a topological group X , the fundamental groupoid $\pi_1 X$ becomes a group object in the category of groupoids [3]. This is also called an internal category in the category of groups [9]. This functor gives an equality of the category of the covering spaces of a topological group X whose underlying space is locally nice and the category of the covering groupoids of $\pi_1 X$ [2, 8].

In this paper we prove that if (X, x_0) is an H -group, then the fundamental groupoid $\pi_1 X$ is a categorical group. This enable us to prove that the category of the covering spaces of an H -group (X, x_0) is equivalent to the category of covering groupoid of the categorical group $\pi_1 X$.

2 Covering Spaces and H -groups

We assume the usual theory of covering maps. All spaces X are assumed to be locally path connected and semi locally 1-connected, so that each path component of X admits a simply connected cover.

Recall that a covering map $p: \tilde{X} \rightarrow X$ of connected spaces is called *universal* if it covers every cover of X in the sense that if $q: \tilde{Y} \rightarrow X$ is another cover of X then there exists a map $r: \tilde{X} \rightarrow \tilde{Y}$ such that $p = qr$ (hence r becomes a cover). A covering map $p: \tilde{X} \rightarrow X$ is called *simply connected* if \tilde{X} is simply connected. Note that a simply connected cover is a universal cover.

Definition 2.1. We call a subset U of X *liftable* if it is open, path connected and U lifts to each cover of X , that is, if $p: \tilde{X} \rightarrow X$ is a covering map, $\iota: U \rightarrow X$ is the inclusion map, and $\tilde{x} \in \tilde{X}$ satisfies $p(\tilde{x}) = x \in U$, then there exists a map (necessarily unique) $\hat{\iota}: U \rightarrow \tilde{X}$ such that $p\hat{\iota} = \iota$ and $\hat{\iota}(x) = \tilde{x}$.

It is easy to see that U is liftable if and only if it is open, path connected and for all $x \in U$, $\iota_* \pi_1(U, x)$ is singleton, where $\pi_1(U, x)$ is the fundamental group of U at the base point x and ι_* is the morphism $\pi_1(U, x) \rightarrow \pi_1(X, x)$ induced by the inclusion map $\iota: U \rightarrow X$. Remark

that if X is a semi locally simply connected topological space, then each point $x \in X$ has a liftable neighbourhood.

Definition 2.2. ([10]) Let X be a topological space. Two covering maps $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ and $q: (\tilde{Y}, \tilde{y}_0) \rightarrow (X, x_0)$ are called equivalent if there is a homeomorphism $f: (\tilde{X}, \tilde{x}_0) \rightarrow (\tilde{Y}, \tilde{y}_0)$ such that $qf = p$.

Definition 2.3. Let $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ be a covering map. The subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ of $\pi_1(X, x_0)$ is called *characteristic group* of p , where p_* is the morphism induced by p .

The following result is known as Covering Homotopy Theorem (Theorem 10.6 in [10]).

Theorem 2.4. Let $p: \tilde{X} \rightarrow X$ be a covering map and Z a connected space. Consider the commutative diagram of continuous maps

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{f}} & \tilde{X} \\ j \downarrow & \tilde{F} \nearrow & \downarrow p \\ Z \times I & \xrightarrow{F} & X \end{array}$$

where $j: Z \rightarrow Z \times I, j(z) = (z, 0)$ for all $z \in Z$. Then there is a unique continuous map $\tilde{F}: Z \times I \rightarrow \tilde{X}$ such that $p\tilde{F} = F$ and $\tilde{F}j = \tilde{f}$

As a corollary of this theorem if the maps $f, g: Z \rightarrow X$ are homotopic, then their respective liftings \tilde{f} and \tilde{g} are homotopic. If $f \simeq g$, there is a continuous map $F: Z \times I \rightarrow X$ such that $F(z, 0) = f(z)$ and $F(z, 1) = g(z)$. So there is a continuous map $\tilde{F}: Z \times I \rightarrow \tilde{X}$ as in Theorem 2.4. Here $p\tilde{F}(z, 0) = F(z, 0) = f(z)$ and $p\tilde{F}(z, 1) = F(z, 1) = g(z)$. By the uniqueness of the liftings we have that $\tilde{F}(z, 0) = \tilde{f}(z)$ and $\tilde{F}(z, 1) = \tilde{g}(z)$. Therefore $\tilde{f} \simeq \tilde{g}$.

Definition 2.5. ([10], p.324) A pointed space (X, x_0) is called an H – group if there are continuous pointed maps

$$m: (X \times X, (x_0, x_0)) \rightarrow (X, x_0), (x, x') \mapsto x \circ x'$$

$$n: (X, x_0) \rightarrow (X, x_0)$$

and pointed homotopies

$$(i) \ m(1_X \times m) \simeq m(m \times 1_X)$$

$$(ii) \ m \ i_1 \simeq 1_X \simeq m \ i_2$$

$$(iii) \ m(1_X, n) \simeq c \simeq m(n, 1_X)$$

where $i_1, i_2: X \rightarrow X \times X$ are injections defined by $i_1(x) = (x, x_0)$ and $i_2(x) = (x_0, x)$, and $c: X \rightarrow X$ is the constant map at x_0 .

In an H -group X , we denote $m(x, x')$ by $x \circ x'$ for $x, x' \in X$.

Let (X, x_0) and (Y, y_0) be H -groups. A continuous map $f: (X, x_0) \rightarrow (Y, y_0)$ such that $f(x \circ x') = f(x) \circ f(x')$ for $x, x' \in X$ is called a *morphism* of H -groups. So we have a category of H -groups denoted by $HGroups$.

Example 2.6. A topological group X with identity e is an H -group. For a topological group X , the group operation

$$m: (X \times X, (e, e)) \rightarrow (X, e)$$

and the inverse map $n: (X, x_0) \rightarrow (X, x_0)$ are continuous; and

- (i) $m(1_X \times m) = m(m \times 1_X)$
- (ii) $m i_1 = 1_X = m i_2$
- (iii) $m(1_X, n) = c = m(n, 1_X)$, where $c: X \rightarrow X$ is the constant map at e .

Theorem 2.7. (Theorem 11.9 in [10]) *If (X, x_0) is a pointed space, then the loop space $\Omega(X, x_0)$ is an H -group.*

Definition 2.8. Let (X, x_0) and (Y, y_0) be H -groups and U an open neighbourhood of x_0 in X . A continuous map $\phi: (U, x_0) \rightarrow (Y, y_0)$ is called a *local morphism* of H -groups if $\phi(x \circ y) = \phi(x) \circ \phi(y)$ for $x, y \in U$ such that $x \circ y \in U$.

Theorem 2.9. *Let (X, x_0) and (\tilde{X}, \tilde{x}_0) be H -groups and $q: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ a morphism of H -groups which is a covering map on the underlying spaces. Let U be an open, path connected neighbourhood of x_0 in X such that U^2 is contained in a liftable neighbourhood V of x_0 in X . Then the inclusion map $\iota: (U, x_0) \rightarrow (X, x_0)$ lifts to a local morphism $\hat{\iota}: (U, x_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ of H -groups.*

Proof: Since V lifts to \tilde{X} , then U lifts to \tilde{X} by $\hat{\iota}: (U, x_0) \rightarrow (\tilde{X}, \tilde{x}_0)$. We now prove that $\hat{\iota}$ is a local morphism of H -groups. By the lifting theorem the map $\hat{\iota}: (U, x_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ is continuous. We have to prove that $\hat{\iota}: (U, x_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ preserves the multiplication. Let $x, y \in U$ such that $x \circ y \in U$. Let a and b be paths from x and y to x_0 in U respectively. By the continuity of

$$m: (X \times X, (x_0, x_0)) \rightarrow (X, x_0)$$

$c = a \circ b$ defined by $c(t) = a(t) \circ b(t)$ for $t \in [0, 1]$ is a path from $x \circ y$ to x_0 . Since $U^2 \subseteq V$, the path c is in V . So the paths a, b and c lift to \tilde{X} . Let \tilde{a}, \tilde{b} and \tilde{c} be the liftings of the paths a, b and c with end points \tilde{x}_0 as chosen above. respectively. Since q is a morphism of H -spaces, we have

$$q(\tilde{c}) = c = a \circ b = q(\tilde{a}) \circ q(\tilde{b}).$$

and

$$q(\tilde{a} \circ \tilde{b}) = q(\tilde{a}) \circ q(\tilde{b}).$$

Since \tilde{c} and $\tilde{a} \circ \tilde{b}$ end at $\tilde{x}_0 \in \tilde{X}$, by the unique path lifting, we have that

$$\tilde{c} = \tilde{a} \circ \tilde{b}$$

By evaluating these paths at $0 \in I$ we have that

$$\widehat{v}(x \circ y) = \widehat{v}(x) \circ \widehat{v}(y).$$

Hence $\widehat{v}: (U, x_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ is a local morphism of H -groups. □

3 Covering Groupoids

A *groupoid* G on O_G is a small category in which each morphism is an isomorphism. Thus G has a set of morphisms, a set O_G of *objects* together with functions $s, t: G \rightarrow O_G$, $\epsilon: O_G \rightarrow G$ such that $s\epsilon = t\epsilon = 1_{O_G}$, the identity map. The functions s, t are called *initial* and *final* point maps respectively. If $a, b \in G$ and $t(a) = s(b)$, then the *product* or *composite* ba exists such that $s(ba) = s(a)$ and $t(ba) = t(b)$. Further, this composite is associative, for $x \in O_G$ the element $\epsilon(x)$ denoted by 1_x acts as the identity, and each element a has an inverse a^{-1} such that $s(a^{-1}) = t(a)$, $t(a^{-1}) = s(a)$, $aa^{-1} = (\epsilon t)(a)$, $a^{-1}a = (\epsilon s)(a)$. The map $G \rightarrow G$, $a \mapsto a^{-1}$, is called the *inversion*.

So a group is a groupoid with only one object.

In a groupoid G for $x, y \in O_G$ we write $G(x, y)$ for the set of all morphisms with initial point x and final point y . We say G is *connected* if for all $x, y \in O_G$, $G(x, y)$ is not empty and *simply connected* if $G(x, y)$ has only one morphism. For $x \in O_G$ we denote the star $\{a \in G \mid s(a) = x\}$ of x by G_x . The *object group* at x is $G(x) = G(x, x)$.

Let G and H be groupoids. A *morphism* from H to G is a pair of maps $f: H \rightarrow G$ and $O_f: O_H \rightarrow O_G$ such that $s \circ f = O_f \circ s$, $t \circ f = O_f \circ t$ and $f(ba) = f(b)f(a)$ for all $(a, b) \in H_t \times_s H$. For such a morphism we simply write $f: H \rightarrow G$.

Definition 3.1. Let $p: \tilde{G} \rightarrow G$ be a morphism of groupoids. Then p is called a *covering morphism* and \tilde{G} a *covering groupoid* of G if for each $\tilde{x} \in O_{\tilde{G}}$ the restriction of p

$$p_x: (\tilde{G})_{\tilde{x}} \rightarrow G_{p(\tilde{x})}$$

is bijective. A covering morphism $p: \tilde{G} \rightarrow G$ is called *connected* if both \tilde{G} and G are connected.

A connected covering morphism $p: \tilde{G} \rightarrow G$ is called *universal* if \tilde{G} covers every cover of G , i.e. if for every covering morphism $q: \tilde{H} \rightarrow G$ there is a unique morphism of groupoids

$\tilde{p}: \tilde{G} \rightarrow \tilde{H}$ such that $q\tilde{p} = p$ (and hence \tilde{p} is also a covering morphism), this is equivalent to that for $\tilde{x}, \tilde{y} \in O_{\tilde{G}}$ the set $\tilde{G}(\tilde{x}, \tilde{y})$ has not more than one element.

A group homomorphism $f: G \rightarrow H$ is a covering morphism if and only if it is an isomorphism.

For any groupoid morphism $p: \tilde{G} \rightarrow G$ and an object \tilde{x} of \tilde{G} we call the subgroup $p(\tilde{G}(\tilde{x}))$ of $G(p\tilde{x})$ the *characteristic group* of p at \tilde{x} .

Example 3.2. [1] If $p: \tilde{X} \rightarrow X$ is a covering map of topological spaces, then the induced fundamental groupoid morphism $\pi_1(p): \pi_1(\tilde{X}) \rightarrow \pi_1(X)$ is a covering morphism of groupoids.

Definition 3.3. Let $p: \tilde{G} \rightarrow G$ be a covering morphism of groupoids and $q: H \rightarrow G$ a morphism of groupoids. If there exists a unique morphism $\tilde{q}: H \rightarrow \tilde{G}$ such that $q = p\tilde{q}$ we just say q lifts to \tilde{q} by p .

We call the following theorem from Brown [1] which is an important result to have the lifting maps on covering groupoids.

Theorem 3.4. Let $p: \tilde{G} \rightarrow G$ be a covering morphism of groupoids, $x \in O_G$ and $\tilde{x} \in O_{\tilde{G}}$ such that $p(\tilde{x}) = x$. Let $q: K \rightarrow G$ be a morphism of groupoids such that K is connected and $z \in O_K$ such that $q(z) = x$. Then the morphism $q: K \rightarrow G$ uniquely lifts to a morphism $\tilde{q}: K \rightarrow \tilde{G}$ such that $\tilde{q}(z) = \tilde{x}$ if and only if $q[K(z)] \subseteq p[\tilde{G}(\tilde{x})]$, where (z) and $\tilde{G}(\tilde{x})$ are the object groups.

Corollary 3.5. Let $p: (\tilde{G}, \tilde{x}) \rightarrow (G, x)$ and $q: (\tilde{H}, \tilde{z}) \rightarrow (G, x)$ be connected covering morphisms with characteristic groups C and D respectively. If $C \subseteq D$, then there is a unique covering morphism $r: (\tilde{G}, \tilde{x}) \rightarrow (\tilde{H}, \tilde{z})$ such that $p = qr$. If $C = D$, then r is an isomorphism.

4 Homotopies of functors and categorical groups

In this section we prove that the functors are homotopic if and only if they are naturally isomorph. For the homotopies of functors we first need the following fact whose proofs are straightforward [1].

Proposition 4.1. Let \mathcal{C}, \mathcal{D} and \mathcal{E} be categories and $F: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ a functor. Then for $x \in Ob(\mathcal{C})$ and $y \in Ob(\mathcal{D})$ we have the induced functors

$$F(x, -): \mathcal{D} \rightarrow \mathcal{E}$$

$$F(-, y): \mathcal{C} \rightarrow \mathcal{E}.$$

We write \mathcal{J} for the simply connected groupoid whose objects are 0 and 1 and non identity morphisms ι and ι^{-1} .

As similar to the homotopies of continuous functions the homotopy of functors are defined as follows [1].

Definition 4.2. Let $f, g: \mathcal{C} \rightarrow \mathcal{D}$ be functors. These functors are called homotopic and written $f \simeq g$ if there is a functor $F: \mathcal{C} \times \mathcal{J} \rightarrow \mathcal{D}$ such that $F(-, 0) = f$ and $F(-, 1) = g$

Proposition 4.3. [1] If the maps $f, g: X \rightarrow Y$ are homotopic, then the induced fundamental groupoid functors $\pi_1 f, \pi_1 g: \pi_1 X \rightarrow \pi_1 Y$ are homotopic.

Definition 4.4. Let $f, g: \mathcal{C} \rightarrow \mathcal{D}$ be two functors. We call f and g are naturally isomorph if there exists a natural isomorphism $\sigma: f \rightarrow g$.

Proposition 4.5. The functors $f, g: \mathcal{C} \rightarrow \mathcal{D}$ are homotopic in the sense of Definition 4.2 if and only if they are naturally isomorph.

Proof: If the functors $f, g: \mathcal{C} \rightarrow \mathcal{D}$ are homotopic there is a functor $F: \mathcal{C} \times \mathcal{J} \rightarrow \mathcal{D}$ such that $F(-, 0) = f$ and $F(-, 1) = g$. Since $(1_x, \iota): (x, 0) \rightarrow (x, 1)$ is an isomorphism in $\mathcal{C} \times \mathcal{J}$ the morphism $F(1_x, \iota): F(x, 0) \rightarrow F(x, 1)$ is an isomorphism in \mathcal{D} where $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. We now define a natural transformation $\sigma: f \rightarrow g$ by $\sigma(x) = F(x, \iota): f(x) \rightarrow g(x)$ for $x \in \mathcal{O}_{\mathcal{C}}$. We now prove that for a morphism $\alpha: x \rightarrow y$ in \mathcal{C} the diagram

$$\begin{array}{ccc} f(x) & \xrightarrow{\sigma(x)} & g(x) \\ f(\alpha) \downarrow & & \downarrow g(\alpha) \\ f(y) & \xrightarrow{\sigma(y)} & g(y) \end{array}$$

is commutative. For this we show that the diagram

$$\begin{array}{ccc} F(x, 0) & \xrightarrow{F(1_x, \iota)} & F(x, 1) \\ F(\alpha, 0) \downarrow & & \downarrow F(\alpha, 1) \\ F(y, 0) & \xrightarrow{F(1_y, \iota)} & F(y, 1) \end{array}$$

is commutative. Since F is a functor

$$\begin{aligned} F(\alpha, 1) \circ F(1_x, \iota) &= F((\alpha, 1) \circ (1_x, \iota)) \\ &= F(\alpha \circ 1_x, 1 \circ \iota) = F(\alpha, 1) \end{aligned}$$

and

$$\begin{aligned} F(1_y, \iota) \circ F(\alpha, 0) &= F((1_y, \iota) \circ (\alpha, 0)) \\ &= F(1_y \circ \alpha, \iota \circ 0) = F(\alpha, 1) \end{aligned}$$

and therefore the latter diagram is commutative. Therefore the functors f and g are naturally isomorph.

Conversely let the functors $f, g: \mathcal{C} \rightarrow \mathcal{D}$ be naturally isomorph. So there is a natural transformation $\sigma: f \rightarrow g$ such that $\sigma_x: f(x) \rightarrow g(x)$ is an isomorphism for each $x \in O_{\mathcal{C}}$ and for $x, y \in O_{\mathcal{C}}$ and $\alpha \in \mathcal{C}(x, y)$ the following diagram is commutative

$$\begin{array}{ccccc} x & & f(x) & \xrightarrow{\sigma(x)} & g(x) \\ \alpha \downarrow & & f(\alpha) \downarrow & & \downarrow g(\alpha) \\ y & & f(y) & \xrightarrow{\sigma(y)} & g(y) \end{array}$$

We now define a homotopy of functors $F: \mathcal{C} \times \mathcal{J} \rightarrow \mathcal{D}$ as follows: Define F on objects by $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for $x \in O_{\mathcal{C}}$. For $x, y, z \in O_{\mathcal{C}}$ consider the following diagram of the morphisms in $\mathcal{C} \times \mathcal{J}$.

$$\begin{array}{ccc} (x, 0) & \xrightarrow{(\alpha, 0)} & (y, 0) \\ (\alpha, \iota) \downarrow & & \uparrow (\alpha, \iota^{-1}) \\ (y, 1) & \xleftarrow{(\alpha, 1)} & (x, 1) \end{array}$$

Define F on these morphisms as follow:

$$\begin{aligned} F(\alpha, 0) &= f(\alpha) \\ F(\alpha, 1) &= g(\alpha) \\ F(\alpha, \iota) &= g(\alpha) \circ \sigma_x \\ F(\alpha, \iota^{-1}) &= f(\alpha) \circ (\sigma_x)^{-1}. \end{aligned}$$

In this way a functor $F: \mathcal{C} \times \mathcal{J} \rightarrow \mathcal{D}$ is defined such that $F(-, 0) = f$ ve $F(-, 1) = g$. Therefore the functors f and g are homotopic. \square

A *group-groupoid* which is also known as *2-group* in literature is a group object in the category of groupoids. The formal definition of a group-groupoid is given in [3] under the name *G-groupoid* as follows:

Definition 4.6. A *group-groupoid* G is a groupoid endowed with a group structure such that the following maps, which are called respectively product, inverse and unit are the mor-

phisms of groupoids:

- (i) $m: G \times G \rightarrow G, (g, h) \mapsto gh$;
- (ii) $u: G \rightarrow G, g \mapsto \bar{g}$;
- (iii) $e: \{\star\} \rightarrow G$, where $\{\star\}$ is singleton.

Here note that the group axioms can be stated as:

- 1. $m(1 \times m) = m(m \times 1)$
- 2. $m i_1 = 1_G = m i_2$
- 3. $m(1, u) = m(u, 1) = e$

where $i_1, i_2: G \rightarrow G \times G$ are injections defined by $i_1(a) = (a, e)$ and $i_2(a) = (e, a)$, and $e: G \rightarrow G$ is the constant map at e .

In the definition of group-groupoid if we take these functors to be homotopic rather than equal, we obtain a kind of definition of categorical group. There are various forms of definitions of categorical group in the literature (see [4] and [7]) and we will use the following one with some weak conditions.

Definition 4.7. Let \mathcal{G} be a groupoid. Let $\otimes: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ and $u: \mathcal{G} \rightarrow \mathcal{G}, a \mapsto \bar{a}$ be functors called respectively product and inverse. Let $e \in O_{\mathcal{G}}$ be an object. If the following conditions are satisfied then $\mathcal{G} = (\mathcal{G}, \otimes, u, e)$ is called a *categorical group*.

- 1. The functors $\otimes(1 \times \otimes), \otimes(1 \times \otimes): \mathcal{G} \times \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ are homotopic.
- 2. The functors $e \otimes 1, 1 \otimes e: \mathcal{G} \rightarrow \mathcal{G}$ defined by $(e \otimes 1)(a) = e \otimes a$ and $(1 \otimes e)(a) = a \otimes e$ for $a \in \mathcal{G}$ are homotopic to the identity functor $\mathcal{G} \rightarrow \mathcal{G}$.
- 3. the functors $\otimes(1, u), \otimes(u, 1): \mathcal{C} \rightarrow \mathcal{C}$ defined by $\otimes(1, u)(a) = a \otimes u(a)$ and $\otimes(u, 1)(a) = u(a) \otimes a$ are homotopic to the constant functor $e: \mathcal{C} \rightarrow \mathcal{C}$.

In this definition if these functors are equal rather than homotopic, then the categorical group is called a *strict categorical group* which is also called group-groupoid or 2-groups.

Note that the product $\otimes: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ is a functor if and only if

$$(b \circ a) \otimes (d \circ c) = (b \otimes d) \circ (a \otimes c)$$

for $a, b, c, d \in \mathcal{G}$ whenever the compositions $b \circ a$ and $d \circ c$ are defined. Since $u: \mathcal{G} \rightarrow \mathcal{G}, a \mapsto \bar{a}$ is a functor when the groupoid composition $b \circ a$ is defined $\overline{b \circ a} = \bar{b} \circ \bar{a}$ and $\overline{1_x} = 1_{\bar{x}}$ for $x \in O_{\mathcal{G}}$.

Proposition 4.8. *If (X, x_0) is an H -group, then the fundamental group $\pi_1 X$ is a categorical group.*

Proof: Since (X, x_0) is an H -group there are continuous maps

$$m: (X \times X, (x_0, x_0)) \rightarrow (X, x_0), (x, y) \mapsto x \circ y$$

$$n: (X, x_0) \rightarrow (X, x_0), x \mapsto \bar{x}$$

and the following homotopies of the maps

$$(i) \ m(1_X \times m) \simeq m(m \times 1_X);$$

$$(ii) \ m \ i_1 \simeq 1_X \simeq m \ i_2;$$

$$(iii) \ m(1_X, n) \simeq c \simeq m(n, 1_X)$$

where $i_1, i_2: X \rightarrow X \times X$ are injection defined by $i_1(x) = (x, x_0)$, $i_2(x) = (x_0, x)$ and $c: X \rightarrow X$ is the constant map at x_0 . From m and n we have the following induced functors

$$\tilde{m} = \pi_1 m: \pi_1 X \times \pi_1 X \rightarrow \pi_1 X$$

and

$$\tilde{n} = \pi_1 n: \pi_1 X \rightarrow \pi_1 X.$$

By Proposition 4.3 from the above homotopies (i), (ii) and (iii), the following homotopies of the functors are obtained

$$(i) \ \pi_1 m(1 \times \pi_1 m) \simeq \pi_1 m(\pi_1 m \times 1)$$

$$(ii) \ \pi_1 m \ \pi_1 i_1 \simeq 1_{\pi_1 X} \simeq \pi_1 m \ \pi_1 i_2$$

$$(iii) \ \pi_1 m(1_{\pi_1 X}, \pi_1 n) \simeq \pi_1 c \simeq \pi_1 m(\pi_1 n, 1_{\pi_1 X}).$$

Therefore $\pi_1 X$ becomes a categorical group. □

Definition 4.9. Let \mathcal{G} and \mathcal{H} be two categorical groups. A morphism of categorical groups is a morphism $f: \mathcal{G} \rightarrow \mathcal{H}$ of groupoids such that $f(a \otimes b) = f(a) \otimes f(b)$ for $a, b \in G$.

So we have a category denoted by *Catgroups* of categorical groups.

Proposition 4.10. If $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ is a morphism of H -groups, then the induced map $\pi_1 p: \pi_1(\tilde{X}) \rightarrow \pi_1(X)$ is a morphism of categorical groups.

Example 4.11. Let (X, x_0) be an H -group. Then we have a slice category

$$HGP\text{Cov}/(X, x_0)$$

of H -group morphisms $f: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ which are covering maps on the underlying spaces. Hence a morphism from $f: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ to $g: (\tilde{Y}, \tilde{y}_0) \rightarrow (X, x_0)$ is a continuous map $p: (\tilde{X}, \tilde{x}_0) \rightarrow (\tilde{Y}, \tilde{y}_0)$ which becomes also a covering map such that $f = gp$.

Similarly we have an other slice category $CatGpCov/\pi_1 X$ of categorical group morphisms $p: \tilde{G} \rightarrow \pi_1(X)$ which are covering morphisms on underlying groupoids.

Theorem 4.12. Let (X, x_0) be an H -group such that the underlying space has a simply connected cover. Then the categories $HGpCov/(X, x_0)$ and $CatGpCov/\pi_1 X$ are equivalent.

Proof: Let $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ be a morphism of H -groups which is a covering map on the spaces. Then by Proposition 4.10 the induced morphism $\pi_1 p: \pi_1 \tilde{X} \rightarrow \pi_1 X$ is a morphism of categorical groups which is a covering morphism of underlying groupoids. So in this way we have a functor

$$\pi_1: HGpCov/(X, x_0) \rightarrow CatGpCov/\pi_1 X.$$

Conversely we define a functor

$$\eta: CatGpCov/\pi_1 X \rightarrow HGpCov/(X, x_0)$$

as follows:

Let $q: \tilde{G} \rightarrow \pi_1 X$ be a morphism of categorical groups which is a covering morphism on the underlying groupoids. Then by 9.5.5 of [1] there is a topology on $\tilde{X} = O_{\tilde{G}}$ and an isomorphism $\alpha: \tilde{G} \rightarrow \pi_1(\tilde{X})$ such that $p = O_q: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ is a covering map and $q = \pi_1(p) \circ \alpha$. Hence categorical group structure on \tilde{G} transports via α to $\pi_1(\tilde{X})$. So we have the morphisms of groupoids

$$\tilde{m}: \pi_1(\tilde{X}) \times \pi_1(\tilde{X}) \longrightarrow \pi_1(\tilde{X})$$

$$\tilde{n}: \pi_1(\tilde{X}) \longrightarrow \pi_1(\tilde{X})$$

such that $\pi_1(p) \circ \tilde{m} = m \circ (\pi_1(p) \times \pi_1(p))$ and $n\pi_1(p) = \pi_1(p)\tilde{n}$. From these morphisms we obtain the maps

$$\tilde{m}: \tilde{X} \times \tilde{X} \longrightarrow \tilde{X}$$

$$\tilde{n}: \tilde{X} \longrightarrow \tilde{X}$$

Since (X, x_0) is an H -group with the maps

$$m: (X \times X, (x_0, x_0)) \rightarrow (X, x_0)$$

$$n: (X, x_0) \rightarrow (X, x_0)$$

we have the following homotopies of pointed maps

- (i) $m(1_X \times m) \simeq m(m \times 1_X)$
- (ii) $m i_1 \simeq 1_X \simeq m i_2$
- (iii) $m(1_X, n) \simeq c \simeq m(n, 1_X)$.

Then by Theorem 2.4 we have the following homotopies

- (i) $\tilde{m}(1_{\tilde{X}} \times \tilde{m}) \simeq \tilde{m}(\tilde{m} \times 1_{\tilde{X}})$
- (ii) $\tilde{m} i_1 \simeq 1_{\tilde{X}} \simeq \tilde{m} i_2$
- (iii) $\tilde{m}(1_{\tilde{X}}, n) \simeq c \simeq \tilde{m}(n, 1_{\tilde{X}})$.

Therefore (\tilde{X}, \tilde{x}_0) is an H -group and $q = Ob(p): (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ is a covering morphism of H -groups.

If $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ is a covering map on underlying spaces, then by 9.5.5 of [1] the topology of \tilde{X} is that of X lifted by the covering morphism $\pi_1 p: \pi_1 \tilde{X} \rightarrow \pi_1 X$ and so $\eta \pi_1 = 1$. Further if $q: \tilde{G} \rightarrow \pi_1 X$ is a morphism of categorical groups, then for the lifted topology on \tilde{X} , \tilde{G} is isomorph to $\pi_1 \tilde{X}$ and so $\eta \pi_1 \simeq 1$. Therefore these functors give an equivalence of the categories. \square

Definition 4.13. Let \mathcal{G} be a categorical group, $e \in O_{\mathcal{G}}$ the base point and let \tilde{G} be just a groupoid. Suppose $p: \tilde{G} \rightarrow \mathcal{G}$ is a covering morphism of groupoids and $\tilde{e} \in O_{\tilde{G}}$ such that $p(\tilde{e}) = e$. We say that the categorical group structure of \mathcal{G} *lifts* to \tilde{G} if there exists a categorical group structure on \tilde{G} with the base point $\tilde{e} \in O_{\tilde{G}}$ such that $p: \tilde{G} \rightarrow \mathcal{G}$ is a morphism of categorical groups

In the following proposition we prove that the liftings of homotopic functors are also homotopic.

Proposition 4.14. Let $p: (\tilde{G}, \tilde{x}) \rightarrow (G, x)$ be a covering morphism of groupoids. Suppose that K is a 1-connected groupoid, i.e, for each $x, y \in O_K$, $K(x, y)$ has only one morphism. Let $f, g: (K, z) \rightarrow (G, x)$ be the morphisms of groupoids such that f and g are homotopics. Let \tilde{f} and \tilde{g} be the liftings of f and g respectively. Then \tilde{f} and \tilde{g} are also homotopic.

Proof: Since the functors f and g are homotopic, there is a functor $F: K \times \mathcal{J} \rightarrow G$ such that $F(-, 0) = f$ and $F(-, 1) = g$. Since K is a 1-connected groupoid by Theorem 3.4 there is a functor $\tilde{F}: (K \times \mathcal{J}, (z, 0)) \rightarrow (\tilde{G}, \tilde{x})$ such that $p\tilde{F} = F$. Hence $p\tilde{F}(-, 0) = F(-, 0) = f$ and $p\tilde{F}(-, 1) = F(-, 1) = g$. So by the uniqueness of the liftings we have that $\tilde{F}(-, 0) = \tilde{f}$ and $\tilde{F}(-, 1) = \tilde{g}$. Therefore \tilde{f} and \tilde{g} are homotopic. \square

Theorem 4.15. Let \tilde{G} be a 1-connected groupoid and \mathcal{G} a categorical group. Suppose that $p: \tilde{G} \rightarrow \mathcal{G}$ is a covering morphism on the underlying groupoids. Let $e \in O_{\mathcal{G}}$ be the base point of \mathcal{G} and $\tilde{e} \in O_{\tilde{G}}$ such that $p(\tilde{e}) = e$. Then the categorical group structure of \mathcal{G} lifts to \tilde{G} .

Proof: Since \mathcal{G} is categorical group, it has the following functors

$$\otimes: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}, (a, b) \mapsto a \otimes b$$

$$u: G \rightarrow G, a \mapsto \bar{a} \text{ and}$$

$$e: \{\star\} \rightarrow G$$

such that the following reduced functors are homotopic:

$$(i) \otimes(1 \times \otimes) \simeq \otimes(\otimes \times 1);$$

$$(ii) \otimes i_1 \simeq \otimes i_2 \simeq 1_G;$$

$$(iii) \otimes(1, n) \simeq c \simeq \otimes(n, 1).$$

Since \tilde{G} is a 1- connected groupoid by Theorem 3.4 the functors \otimes and u lift respectively to the morphisms of groupoids

$$\tilde{\otimes}: (\tilde{G} \times \tilde{G}, (\tilde{e}, \tilde{e})) \rightarrow (\tilde{G}, \tilde{e})$$

and

$$\tilde{u}: (\tilde{G}, \tilde{e}) \rightarrow (\tilde{G}, \tilde{e})$$

By Proposition 4.14 we have the following homotopies of the functors:

$$(i) \tilde{\otimes}(1 \times \tilde{\otimes}) \simeq \tilde{\otimes}(\tilde{\otimes} \times 1);$$

$$(ii) \tilde{\otimes} i_1 \simeq \tilde{\otimes} i_2 \simeq 1_{\tilde{G}};$$

$$(iii) \tilde{\otimes}(1, \tilde{u}) \simeq c \simeq \tilde{\otimes}(\tilde{u}, 1). \quad \square$$

As a result of Theorem 4.15 we obtain the following corollary.

Corollary 4.16. Let (X, x_0) be an H -group and $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ a covering map. If \tilde{X} is a simply connected topological space, then H -group structure of (X, x_0) lifts to (\tilde{X}, \tilde{x}_0) , i.e, (\tilde{X}, \tilde{x}_0) is an H -group and $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ is a morphism of H -groups.

Proof: Since $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ is a covering map, the induced morphism $\pi_1 p: \pi_1 \tilde{X} \rightarrow \pi_1 X$ is a covering morphism of groupoids. Since (X, x_0) is an H -group by Proposition 4.8 $\pi_1 X$ is a categorical group and since \tilde{X} is simply connected the fundamental groupoid $\pi_1 \tilde{X}$ is a simply connected groupoid. So by Proposition 4.14 the categorical group structure of $\pi_1 X$ lifts to $\pi_1 \tilde{X}$. So we have the groupoid morphisms

$$\tilde{m}: \pi_1(\tilde{X}) \times \pi_1(\tilde{X}) \longrightarrow \pi_1(\tilde{X})$$

$$\tilde{n}: \pi_1(\tilde{X}) \longrightarrow \pi_1(\tilde{X})$$

such that $\pi_1(p) \circ \tilde{m} = m \circ (\pi_1(p) \times \pi_1(p))$ and $n\pi_1(p) = \pi_1(p)\tilde{n}$ and so the maps

$$\tilde{m}: \tilde{X} \times \tilde{X} \longrightarrow \tilde{X}$$

$$\tilde{n}: \tilde{X} \longrightarrow \tilde{X}$$

Since (X, x_0) is an H -group, we have the homotopies

$$(i) \ m(1_X \times m) \simeq m(m \times 1_X)$$

$$(ii) \ m \ i_1 \simeq 1_X \simeq m \ i_2$$

$$(iii) \ m(1_X, n) \simeq c \simeq m(n, 1_X)$$

and so by Theorem 2.4 the homotopies

$$(i) \ \tilde{m}(1_X \times \tilde{m}) \simeq \tilde{m}(\tilde{m} \times 1_X)$$

$$(ii) \ \tilde{m} \ i_1 \simeq 1_X \simeq \tilde{m} \ i_2$$

$$(iii) \ \tilde{m}(1_X, n) \simeq c \simeq \tilde{m}(n, 1_X).$$

Therefore (\tilde{X}, \tilde{x}_0) is an H -group and $q = Ob(p): (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ is a covering morphism of H -groups.

Therefore (\tilde{X}, \tilde{x}_0) becomes an H -group as required. \square

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